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University of Oregon; Fall 2007

NAME*:** ***SOLUTIONS

Physics 351 – Vibrations and Waves

MIDTERM EXAMINATION

Format: Closed book; no calculators. Four problems.
Time: 50 minutes. (Suggested times: #*1*, 5 min; #*2*, 10 min., #*3* & #*4*, each 15-20 min.)
Write clearly!

You may make use of the following if needed:

For the steady state response of a damped oscillator driven at angular frequency ω:

The amplitude (A) and phase offset (δ) are given by:

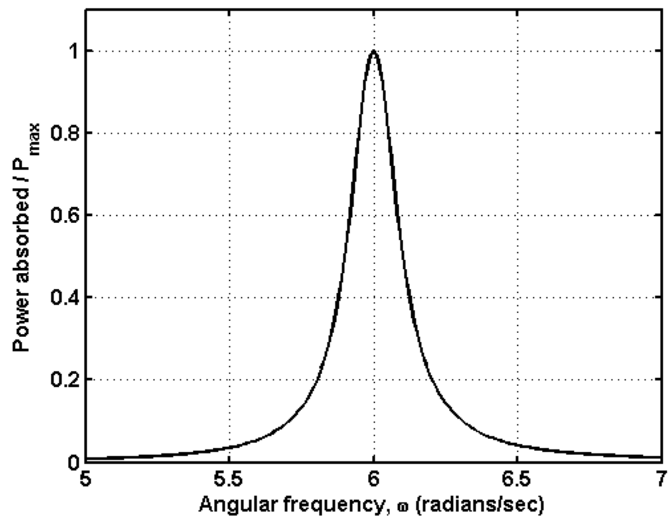
$$A(\omega) = \frac{F_0 / m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \left(\gamma\omega\right)^2\right]^{1/2}}; \qquad \tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

The power absorbed (averaged over one cycle): $\bar{P}(\omega) = \frac{F_0^2 \omega_0}{2kQ}\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{-1}$

As usual:

m is the mass; k is the spring constant.; γ is the damping factor ("$=b/m$")
ω_0 is the angular frequency of the undamped system.
F_0 is the amplitude of the driving force

(1, 7 pts. total**) A driven oscillator.** The power-absorption versus angular frequency, $\bar{P}(\omega)$, for a driven oscillator is measured and plotted (see graph, right).

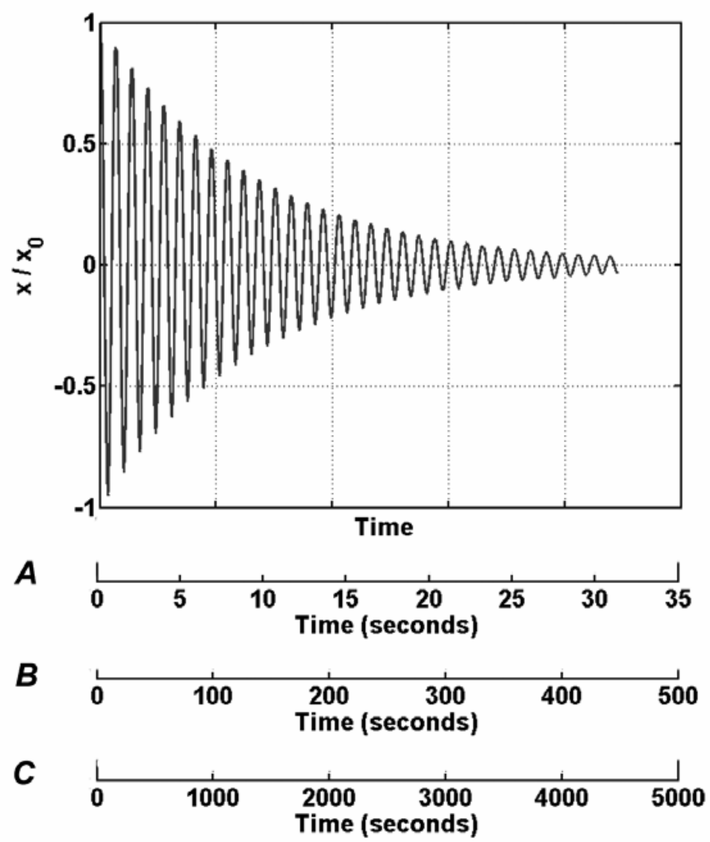


(a, 2 pts.**)** If the Q-factor of the system is increased, will the peak become narrower or wider? [*Just provide a one-word answer.*]

Answer: *Narrower.*

(b, 5 pts.**)** The driving is turned off and the oscillations freely decay from an initial value x_0. I've plotted x(t) – **see below** – but due to an accident with scissors I chopped the numbers off the time scale of the graph. **Which of the time scales (A, B, C) shown below the plot is the correct one for this system? Very briefly explain your answer**.

Answer: A. *The width (FWHM) of the* $\bar{P}(\omega)$ *curve is about 0.2 rad/sec. We know that the FWHM is equal to* γ*, the damping factor, and that the amplitude of free oscillations decays as* $e^{-\gamma t/2}$*. Therefore the 1/e decay time is* $2/\gamma \approx 10$ *seconds, corresponding to scale A.*



(2, 7 pts.**) Oscillation in a sinusoidal potential.** Consider a particle of mass *m* subject to a potential energy $U(x) = U_0 \cos\left(2\pi \frac{x}{\lambda}\right)$, where U_0 and λ are constants and x is position. Determine the angular frequency (ω) of small oscillations about any equilibrium position.

Answer: *The equilibrium positions are the minima of U(x), which occur at* x_0 = $\lambda/2$, $3\lambda/2$, $5\lambda/2$, ... *If this isn't obvious, note that*

$\frac{dU}{dx} = -U_0 \frac{2\pi}{\lambda} \sin\left(2\pi \frac{x}{\lambda}\right) = 0$ *at x=integer** $\lambda/2$, *and* $\frac{d^2U}{dx^2} = -U_0 \left(\frac{2\pi}{\lambda}\right)^2 \cos\left(2\pi \frac{x}{\lambda}\right) > 0$ *only at the odd-integer multiples.*

All the equilibrium points are equivalent.

Taylor expanding U(x) around x_0,

$U(x) = U(x_0) + \left.\frac{dU}{dx}\right|_{x_0} (x - x_0) + \frac{1}{2} \left.\frac{d^2U}{dx^2}\right|_{x_0} (x - x_0)^2 + \ldots$, *which for* x_0 *being an equilibrium point becomes*

$$U(x) = U(x_0) + \frac{1}{2} \left.\frac{d^2U}{dx^2}\right|_{x_0} (x - x_0)^2 + \ldots$$

$U(x) \approx U(x_0) + \frac{1}{2} U_0 \left(\frac{2\pi}{\lambda}\right)^2 (x - x_0)^2$, *limiting ourselves to small* $(x-x_0)$ *so that we can neglect higher-order terms.*

As we are well aware, having done similar exercises before, this looks just like the potential energy function for a spring: $U(x) = \frac{1}{2}"k"x^2$, *with x being <u>deviation from equilibrium</u> (i.e. like* $(x-x_0)$ *above) and spring constant k, except for an irrelevant offset* $(U(x_0))$. *Here,* $"k" = U_0 \left(\frac{2\pi}{\lambda}\right)^2$. *Therefore the oscillation frequency*

$$\boxed{\omega = \sqrt{k/m} = \frac{2\pi}{\lambda}\sqrt{\frac{U_0}{m}}}$$

(3, 8 pts.**) A damped, oscillating sphere.** Consider a sphere of radius r and density ρ attached to a spring of spring constant k. Damping is due to a drag force $F = -Crv$, where v is the (1-dimensional) velocity and C is a constant that depends on viscosity and other parameters.

(a, 2 pts.**)** Determine the differential equation of motion of the sphere about its equilibrium position; express your equation in terms of r, ρ, C, k, and numerical constants only. (A suggestion that may make (***b***) easier: write your differential equation as "$\ddot{x} +$ other terms = 0".)

(b, 6 pts.**)** I choose a spring such that the system is critically damped. Then, Mr. K. wanders in and replaces the sphere with one of the same density but a larger radius (leaving everything else unchanged). Is the new system overdamped, underdamped, or critically damped? (*Hint*: It isn't necessary to first determine k, but if stuck you may find it useful to do so.)

(a) The forces acting on the mass are the spring force, $-kx$ and the drag force $-Cr\dot{x}$; these equal $m\ddot{x}$ (Newton). The mass of the sphere is ρ times its volume, $\frac{4}{3}\pi r^3$. Therefore $\boxed{\rho \frac{4}{3}\pi r^3 \ddot{x} = -kx - Cr\dot{x}}$, which we can rewrite for later convenience as

$$\boxed{\ddot{x} + \frac{3C}{\rho 4\pi r^2}\dot{x} + \frac{3k}{\rho 4\pi r^3}x = 0}.$$

(b) I'll write two ways of solving this. Both make use of the fact that the above differential equation is that of our usual damped oscillator, with

$$\gamma = \frac{3C}{\rho 4\pi r^2} \text{ and } \omega_0 = \sqrt{\frac{3k}{\rho 4\pi r^3}}.$$

(***Method 1***) The damping regime (over-, under-, critical) depends on the ratio $\frac{\gamma}{2\omega_0}$. How does this depend on r, if all other factors are constant? From the above relations $\gamma \propto r^{-2}$ and $\omega_0 \propto r^{-3/2}$, so $\frac{\gamma}{2\omega_0} \propto r^{-1/2}$. Therefore if we start off being critically damped and increase r, $r^{-1/2}$ will decrease and $\frac{\gamma}{2\omega_0}$ will decrease; therefore we will have an **underdamped** system.

(***Method 2***) At critical damping, $\gamma = 2\omega_0$, from which we can determine the k that satisfies this for r_0, the initial radius. Using the above relations:

$$\gamma = 2\omega_0 \rightarrow \frac{3C}{\rho 4\pi r_0^2} = 2\sqrt{\frac{3k}{\rho 4\pi r_0^3}} \rightarrow k = \frac{3C^2}{16\rho\pi r_0}.$$

Changing r, what is $\frac{\gamma}{2\omega_0}$? The ratio

$$\frac{\gamma}{2\omega_0} = \frac{3C}{\rho 4\pi r^2 2}\sqrt{\frac{\rho 4\pi r^3}{3k}} = \frac{3C}{\rho 4\pi r^2 2}\sqrt{\frac{\rho 4\pi r^3 16\rho\pi r_0}{3\times 3C^2}} (\text{substituting } k) = \sqrt{\frac{r_0}{r}}.$$

Therefore if $r > r_0$, $\frac{\gamma}{2\omega_0} < 1$ and the new system is **underdamped.**

(4, 9 pts.**) Two monkeys.** If bent, a tree branch acts like a spring, i.e. pulling back with a force $-kx$ in response to a vertical deflection x. (See the figure.). Friction within the wood is the dominant damping and is independent of whatever mass is hung from it. (I.e. "b" in $F=-b\dot{x}$ is always the same.) **One** monkey (of mass m_1) grabs the branch and hangs from it. Hiding in the bushes (after installing energy sensors in the tree) we observe that

(*i*) The monkey oscillates *many* times up and down before being appreciably damped.

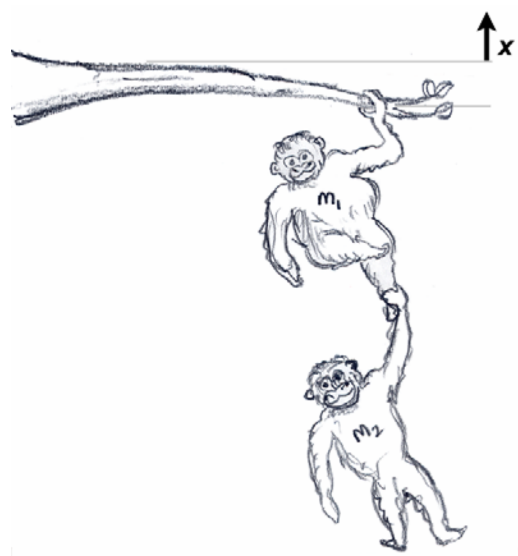
Then, a **second** monkey (of mass m_2) grabs onto the first monkey (see figure). We observe that:

(*ii*) The two monkeys together also oscillate *many* times before being appreciably damped.

(*iii*) The equilibrium branch deflection to which the system settles is **three times** greater than the deflection of the branch with just one monkey.

(*iv*) The energy of the oscillations relative to the initial energy after **one** cycle of oscillation (i.e. $E(T)/E_0$, where T is the period) is **half** as large for the one-monkey system as for the pair of monkeys.

Based on these observations, **determine m_1 and m_2 in terms of k and b.** (*Hint*: Observation (*iii*) alone is sufficient to reveal m_1/m_2.) If you don't have time to simplify the math, put your answer into a form from which one *could* solve for the masses in terms of other, known, parameters by simple algebra; you'll lose at most a point.



Answer: From (i) and (ii), we know we have a weakly damped system.

From (iii): At equilibrium, the gravitational force equals the "spring" force, so: $mg = kx$.

One monkey: $m_1 g = kx_1$

Two monkeys: $(m_1+m_2)g = kx_2$.

We're told that $x_2 = 3x_1$. Dividing the above expressions, $\frac{x_2}{x_1} = \frac{m_2+m_1}{m_1} = 3$, so $\frac{m_2}{m_1}+1=3$, so $\boxed{m_2 = 2m_1}$.

Now we consider the decay of the energy of oscillations. <u>We know that in general</u>, $E(t) = E_0 e^{-\gamma t}$, where $\gamma = b/m$. At one cycle, $t = T = \frac{2\pi}{\omega} \approx \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$, using the fact that damping is weak, and noting $\omega_0 = \sqrt{k/m}$. Therefore at $t=T$, $E(T) = E_0 \exp\left(-2\pi \frac{b}{m}\sqrt{\frac{m}{k}}\right) = E_0 \exp\left(-2\pi b\sqrt{\frac{1}{km}}\right)$.

One monkey: $E_1(T) = E_{0,1}\exp\left(-2\pi b\sqrt{\frac{1}{km_1}}\right)$

Two monkeys: $E_2(T) = E_{0,2}\exp\left(-2\pi b\sqrt{\frac{1}{k\left(m_1+m_2\right)}}\right)$

We're told (obs. (iv)) that $\frac{E_1(T)}{E_{0,1}} = \frac{1}{2}\frac{E_2(T)}{E_{0,2}}$, therefore:

$\exp\left(-2\pi b\sqrt{\frac{1}{km_1}}\right) = \frac{1}{2}\exp\left(-2\pi b\sqrt{\frac{1}{k\left(m_1+m_2\right)}}\right)$. Taking logarithms:

$$-2\pi b\sqrt{\frac{1}{km_1}} = \ln\left(\frac{1}{2}\right) - 2\pi b\sqrt{\frac{1}{k\left(m_1+m_2\right)}}$$

$$\frac{-2\pi b}{\sqrt{km_1}} = -\ln\left(2\right) - \frac{2\pi b}{\sqrt{k\left(m_1+m_2\right)}}$$

$\frac{2\pi b}{\sqrt{k}}\left(\frac{1}{\sqrt{m_1+m_2}} - \frac{1}{\sqrt{m_1}}\right) = -\ln(2)$. From above, $m_2 = 2m_1$, so:

$\frac{1}{\sqrt{m_1}}\left(1-\frac{1}{\sqrt{3}}\right) = \ln(2)\frac{\sqrt{k}}{2\pi b}$, from which:

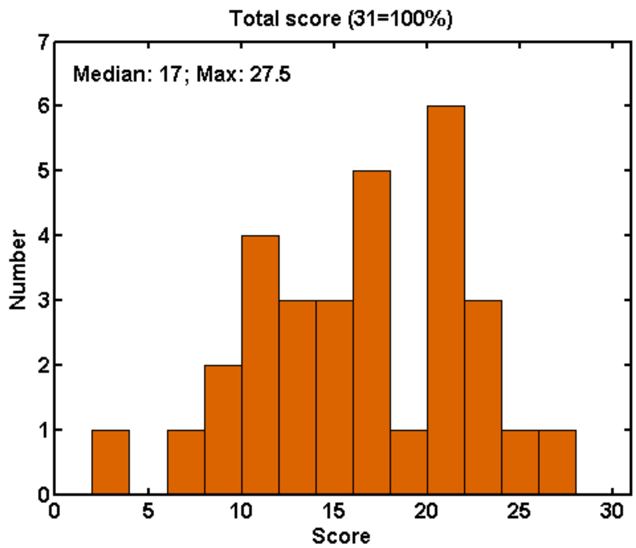
$\boxed{m_1 = \left(\frac{2\pi b}{\ln(2)\left(1-\frac{1}{\sqrt{3}}\right)}\right)^2\frac{1}{k}}$, and $\boxed{m_2 = 2m_1}$.

ADDITIONAL COMMENTS ABOUT THE MIDTERM – RP

Overall: I was expecting a mean score around 24; the actual values of the mean and median were both 17 (out of 31). As mentioned in class, I am disappointed. The most worrying aspect is the poor performance on Problem 2 – involving oscillations in an "arbitrary" potential, which we've seen before and whose importance I have stressed.

I've had good conversations with several of you, and based on earlier impressions think that most of you are bright enough and hard-working enough to do considerably better than this exam would indicate. (And, of course, several people did do very well on the exam.) I hope you'll study the solutions, and see me about any difficulties. Also, you may wish to think about your "test taking" philosophies. In many cases, I want to see that you can distill a setup or concept into its physical & mathematical essence – I care only slightly about the algebra of the solution, since I'm fairly sure that if you have time you can work through algebra properly. If you understand what's going on, this "distillation" should not be a long or painful process, and you should not let worries that you won't have time to work out algebra somehow derail you from thinking clearly about the physics! This is especially relevant to Problem 4 – many people have said they "only had 5 minutes" to work on it, but 5 minutes is more than enough time to set up the solution.

You may find the comments on particular problems below useful.



Problem 1.

We derived $\overline{P}(\omega)$ in class and in the text, and I stressed **repeatedly** (at least 3 times) that the most important (and the most interesting) thing about it is that the width of the curve equals γ, the damping factor – in other words, the **resonant response** and the **free decay** of an oscillator are intimately connected. From this fact alone, you can simply "read off" that the width ≈ 0.2 rad/sec, so decay time is roughly the reciprocal of this, indicating timescale "A" rather than "B" or "C".

Perhaps the weakest answer is to simply state that decay must be "fast," without a number. What does this mean?! If one of the answer choices had been a scale from 0 to 10 microseconds, should we choose it?

Problem 2.

From Problem Set 2:

(3, 7 pts.**) A rolling ball.** A ball of mass *m* moves in a one-dimensional landscape of hills and valleys with height *h* as a function of lateral position *x* being given by the function $h(x) = \frac{a}{\sqrt{x}} + bx$, where *a* and *b* are positive constants. The system is in a uniform gravitational field with acceleration *g*, as usual. ...

From the midterm study guide:

- Know how to analyze the frequency of oscillation about equilibrium for objects subject to arbitrary potential energy functions.

It amazes me that anyone would be surprised by this exam question, or incapable of doing it. And, of course, I'm surprised that some people are still hazy on the concept of Taylor Series. Also, some you seem unclear about what a "cosine" looks like.

Problem 3.

From the midterm study guide:

- *Damped oscillators.* ... Know (remember) conditions for over-, under-, and critical-damping..

What determines the conditions for over-, under-, and critical damping? The relative magnitudes of γ and ω_0. There's nothing else. So: examine the dependence on "r" of $\frac{\gamma}{2\omega_0}$. There are several ways to approach this (see the Solutions for two).

Problem 4.

There are several key things this problem tests.

First: Can you turn a conceptual observation (e.g. "The equilibrium branch deflection to which the system settles is **three times** greater than the deflection of the branch with just one monkey") into a mathematical statement?

Second: Do you understand what "equilibrium" means?

Third: Can you relate our general expressions concerning the decay of oscillator energy to this context?

Just getting the "equilibrium" relation right is an easy 3 points.
Recall, from Problem Set 2:

(2, 4 pts. total**) A vertical mass & spring setup.** We've considered in class a horizontal setup of a mass on a spring, in which we could neglect the role of gravity. Now, consider a vertical setup. A mass *m* hangs from a massless spring of stiffness *k*, in a gravitational field whose acceleration is *g*. Gravity leads to extension of the spring – at static equilibrium (i.e. if the mass isn't moving), the gravitational force exactly balances the spring's restoring force, and the spring is extended by some length. Now consider a moving mass: Show that vertical oscillations of the mass <u>about the equilibrium point</u>...

About the rest: Yes, I know you may have been running out of time, but simply turning the energetic observation into a mathematical relation, given our understanding of how energy decays (e.g. PS4 no. 6) without solving it at all, would have scored most of the points of the problem. (This isn't an algebra test!) You *should* be able to quickly "distill" a statement into a mathematical relation – it takes little time, just understanding. I admit that I'm puzzled about how best to convey this understanding. Part of the reason I write verbose problem sets is for this exact reason – perhaps they're not verbose enough...